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ON THE CALCULATION OF LIMIT AND BIFURCATION POINTS IN STABILITY PROBLEMS OF ELASTIC SHELLS†,‡

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Abstract—Numerical methods for the computation of singular points of nonlinear equations $G(u, \lambda, \mu) = 0$ are discussed, where λ and μ are real parameters. Simple and double limit points are treated in some detail and numerical algorithms are presented and applied to elastic shell stability problems. The case of simple symmetry breaking bifurcation points is also treated with applications to nonsymmetric bifurcation from axisymmetric states of deformation of shells of revolution.

1. INTRODUCTION

The numerical analysis of singular points of nonlinear functional equations has been the subject of a series of recent contributions in the literature. It is the purpose of this paper to present some selected results in a form directly applicable to the calculation of limit and bifurcation points, with particular application to stability problems of elastic shells. This amounts to giving a more detailed analysis of some of the methods and to supplement them appropriately, in order to convert some theoretical results of singular point theory into numerical algorithms that work.

The mathematical theory of limit and bifurcation points is usually developed in an abstract functional analysis setting. But numerical analysts have found that many theoretical results are directly applicable to the computation of certain simple types of singular points and of branches of solutions emanating from bifurcation points. Formerly, numerical stability analyses often gave unreliable and inaccurate answers because of the numerical difficulties encountered near singular points. We wish to show that these difficulties can be overcome by making appropriate use of recent results of singularity theory. The price to be paid is that so-called "extended systems" of equations have to be solved numerically, which can be done quite efficiently using standard computer software, as will be shown in what follows.

2. EXAMPLE: BUCKLING OF SPHERICAL SHELLS

It is well known in the theory of stability of elastic shells that basically two types of instabilities may occur: limit points, usually connected with snap-through phenomena, and bifurcation points. Both types can be illustrated by the problem of a shallow spherical shell under uniform pressure p, with $\mu = 2(H/h)\sqrt{12(1 - \nu^2)}$ describing the shell geometry [1]. Let v be the deformed volume and consider axisymmetric deformations. Then we have a nonlinear monotone load–deflection curve for $\mu < \mu_0$, and an s-shaped curve with two limit (turning) points for $\mu > \mu_0$. In Fig. 1, the points A and B are examples of "simple" limit points. At the transition between the two types of p–v-curves, we find a "double" limit point C at $\mu = \mu_0$.

The bifurcation type instability occurs, for instance, if new branches of nonsymmetric normal deflections of the form $w_n(r) \cos n\theta$, $n = 1, 2, \ldots$ take off from the axisymmetric p–v-relation at certain critical values p_n. The points D, E in Fig. 1 are "simple" bifurcation points while F is a "double" bifurcation point, where two nonsymmetric branches intersect the "primary branch" at the load p_F.

† Dedicated to my teacher and friend Eric Reissner, who has had a decisive influence on my work, on the occasion of his seventieth anniversary.

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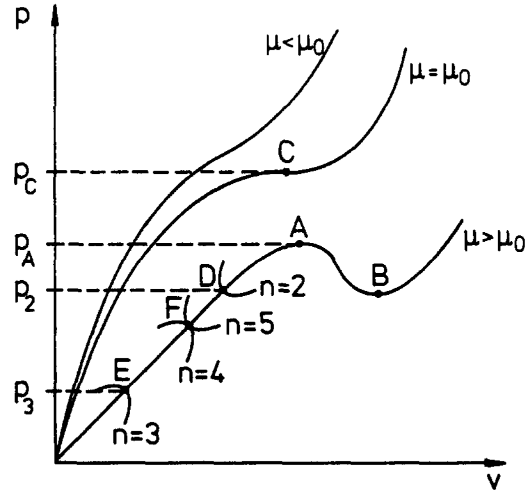

Fig. 1. Load deflection diagram for a shallow spherical shell under uniform pressure: axisymmetric limit points A, B, C (snap buckling) and symmetry breaking bifurcation buckling.

A nonlinear boundary value problem must be solved in order to find a point on the axisymmetric p–v-curve of Fig. 1. This can be done by the standard Newton method, provided the solutions are isolated, which is a basic requirement for any numerical algorithm. However, at a limit point the solution is no longer isolated, hence numerical difficulties must be expected. In order to locate a limit point with some accuracy by interpolation, a large number of closely spaced points on the p–v-curve should be computed, which is both costly and difficult because of the near singularity of the boundary value problem near a limit point (see Fig. 2).

Similarly, nonsymmetric bifurcation points can be found by interpolation, if sufficiently many points are computed near the critical loads p_n [2]. These procedures of indirect calculation of limit and bifurcation points are unsatisfactory, they involve a good deal of trial and error strategy, and they yield little information on the accuracy, in particular in the case of limit points.

3. THE COMPUTATION OF SIMPLE LIMIT POINTS

The essence of the more recent approach to compute singular points directly and accurately consists in deriving from the given equations a new system of equations called the extended system whose solution at the limit or bifurcation point is isolated.

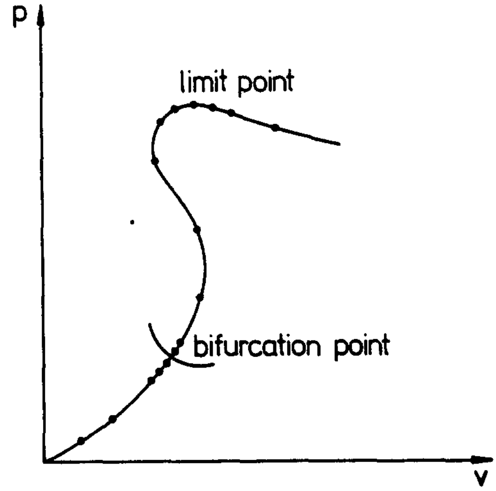

Fig. 2. Indirect approximate calculation of limit and bifurcation points by interpolation.

The Newton method is then applicable, provided that close enough starting values for the iteration for the extended system are found. The latter point is usually not adequately treated in papers on numerical methods. However, in the present context it is crucial to devise methods that will produce appropriate starting values.

We begin with some formal definitions and known results from bifurcation theory, employing standard notation (see [3–7]). The basic equations and boundary conditions are denoted by $G(u, \lambda) = 0$, where G is a nonlinear mapping $G: \mathcal{B} \times \mathcal{R} \to \mathcal{B}$, $\mathcal{B}$ a Banach space. The solution set is

$$S_G = \{(u, \lambda) \mid G(u, \lambda) = 0, \quad u \in \mathcal{B}, \quad \lambda \in \mathcal{R}\}.$$

Assuming that all Fréchet derivatives G_u, G_λ, $G_{uu}, \ldots$ of G are continuous, and denoting by $\langle v, u \rangle$ the dual pairing between elements $u \in \mathcal{B}$ and $v \in \mathcal{B}'$ ($\mathcal{B}'$ = dual space of $\mathcal{B}$), we have

Definition 1. $(\bar{u}, \bar{\lambda})$ is called a regular point of S_G if $G_u^{-1}(\bar{u}, \bar{\lambda})$ exists, otherwise it is called a singular (exceptional) point.

Definition 2. A singular point $(u_0, \lambda_0) \in S_G$ is called a limit (turning) point with respect to λ, if

(a) $\dim \operatorname{Ker} G_u(u_0, \lambda_0) = 1$, $\operatorname{Ker} G_u = \{\alpha\phi_0 \mid \alpha \in \mathcal{R}\}$,
(b) $\operatorname{codim} \operatorname{Range} G_u(u_0, \lambda_0) = 1$, $\operatorname{Range} G_u = \{v \mid \langle \psi_0, v \rangle = 0\}$,
(c) $G_\lambda(u_0, \lambda_0) \notin \operatorname{Range} G_u(u_0, \lambda_0)$,

where $\phi_0 \in \mathcal{B}$, $\phi_0 \neq 0$ and $\psi_0 \in \mathcal{B}'$, $\psi_0 \neq 0$. Condition (b) implies $G_u(u_0, \lambda_0)'\psi_0 = 0$, where L' is the adjoint of a linear operator $L: \mathcal{B} \to \mathcal{B}$. If $(\bar{u}, \bar{\lambda})$ is regular, $h = 0$ is the only solution of $G_u(u_0, \lambda_0)h = 0$, otherwise a nontrivial solution $h \neq 0$ exists. It is condition (c) that distinguishes a limit point from a bifurcation point.

It is well known that near a limit point (u_0, λ_0) S_G may be represented by $u(s)$, $\lambda(s)$, for $|s - s_0| \leq \delta$, s real, such that

$$\begin{aligned} &u(s_0) = u_0, \quad \lambda(s_0) = \lambda_0, \quad |\dot{\lambda}(s)| + \|\dot{u}(s)\| > 0 \\ &\dot{u}(s_0) = \phi_0, \quad \dot{\lambda}(s_0) = 0, \quad u(s) = s\phi_0 + v(s), \end{aligned} \tag{1}$$

where $v \in V_0$ and $\mathcal{B} = \operatorname{Ker} G_u(u_0, \lambda_0) \oplus V_0$.

Definition 3. A limit point $(u_0, \lambda_0) \in S_G$ is called simple if $\ddot{\lambda}(s_0) \neq 0$, otherwise it is nonsimple; if $\ddot{\lambda}(s_0) = 0$, $\dddot{\lambda}(s_0) \neq 0$, (u_0, λ_0) is called a double limit point.

We now introduce the extended system of $G(u, \lambda) = 0$ at a limit point by

$$\left.\begin{aligned} G(u, \lambda) &= 0 \\ G_u(u, \lambda)h &= 0 \\ l(h) &= 1 \end{aligned}\right\} \hat{G}(u, h, \lambda) = 0 \tag{2}$$

l is a functional with the properties $l(0) = 0$, and $l(h) \neq 0$ implying $h \neq 0$. If l is linear, we have $l(h) = \langle l, h \rangle$, $l \in \mathcal{B}'$.

The significance of $\hat{G} = 0$ is that calculation of $\hat{G}_v$, with $v := (u, h, \lambda)$, shows that (a)–(c) of Definition 2, together with eqn (1) and the condition $\ddot{\lambda}(s_0) \neq 0$ imply that $\hat{G}_v^{-1}$ exists, and hence $\hat{G}(u, h, \lambda) = 0$ has isolated solutions. In recent papers, Moore, Spence and Werner [8, 9] proved that the converse also holds. Thus one has

THEOREM A. $(u_0, \lambda_0) \in S_G$ is a simple limit point if and only if

$$\operatorname{Ker} \hat{G}_v(u_0, \phi_0, \lambda_0) = \{0\}, \quad \operatorname{Range} \hat{G}_v(u_0, \phi_0, \lambda_0) = \mathcal{B} \times \mathcal{B} \times \mathcal{R}.$$

The explicit computation of $\hat{G}_v$ yields

$$\hat{G}_v(u_0, \phi_0, \lambda_0)\Phi = \begin{pmatrix} G_u^0 w + G_\lambda^0 \mu \\ G_{uu}^0 h w + G_u^0 k + G_{u\lambda}^0 h \mu \\ \langle l, k \rangle \end{pmatrix}, \quad \Phi = \begin{pmatrix} w \\ k \\ \mu \end{pmatrix} \tag{3}$$

where a superscript 0 indicates evaluation at (u_0, λ_0), that is, $G_u^0 = G_u(u_0, \lambda_0)$, etc. In classical notation w, k, μ are the variations δu, δh, $\delta\lambda$, respectively. Furthermore, we have from [8]:

THEOREM B. If $(u_0, \lambda_0) \in S_G$ is a nonsimple limit point, then dim Ker $\hat{G}_v^0 = 1 =$ codim Range $\hat{G}_v^0$, and

$$\begin{aligned} \text{Ker } \hat{G}_v(u_0, \phi_0, \lambda_0) &= \{\alpha\Phi_0 \mid \alpha \in \mathfrak{R}\}, \quad \Phi_0 = (\phi_0, u_1, 0) \\ G_u^0 u_1 &= -G_{uu}^0\phi_0\phi_0, \qquad \langle l, u_1\rangle = 0. \end{aligned}$$

This theorem implies that eqn (2) is not a suitable system for computing double limit points. This case will be discussed in the next section. For simple limit points, Theorem A implies that eqn (2) can be solved by Newton's method.

In applications, one usually computes a sequence of solutions of $G(u, \lambda) = 0$ on a branch C_1 of S_G where one expects to find limit points. Suppose $(\bar{u}, \bar{\lambda})$ is a regular point on C_1 close to a simple limit point (u_0, λ_0). In order to switch from solving $G(u, \lambda) = 0$ to solving the extended system $\hat{G}(u, h, \lambda) = 0$, we need a starting value $\bar{h}$ for h in order to apply Newton's method to $\hat{G}(u, h, \lambda) = 0$, while $u = \bar{u}$ and $\lambda = \bar{\lambda}$ may serve as starting values for u and λ, respectively. $\bar{h}$ should satisfy the last two equations of (2) approximately. The method proposed here [10] is to introduce an inhomogeneity $c^* \in \mathfrak{B}$ with $c^* \neq 0$. Then the equation

$$G_u(\bar{u}, \bar{\lambda})h^* = c^* \tag{4}$$

has a unique solution $h^* \neq 0$. Now set $h_0 := \alpha h^*$, $\alpha \in \mathfrak{R}$, and determine α from $l(h_0) = l(\alpha h^*) = 1$. It follows that h_0 satisfies

$$G_u(\bar{u}, \bar{\lambda})h_0 = c, \quad l(h_0) = 1, \quad \text{with} \quad c = \alpha c^* \tag{5}$$

and we have $c \to 0$ as $\lambda \to \lambda_0$. The problem (4) is linear and is solved together with $G(u, \lambda) = 0$. As soon as $\| c \|$ in eqn (5) decreases significantly along C_1, a singular point is approached and we then switch to solving eqn (2), taking $\bar{u}$, λ, h_0 as starting values for the Newton iteration. The method is simple and generally applicable.

A different method for computing starting values has been proposed in [11]. However, it is restricted to the case where G represents a boundary value problem for ordinary differential equations (see the discussion in [12]).

Remark. The condition that the equation for u_1 in Theorem B is solvable can be written as $\langle\psi_0, G_{uu}^0\phi_0\phi_0\rangle = 0$. Thus one has the following characterization of simple limit points

$$\langle\psi_0, G_{uu}(u_0, \lambda_0)\phi_0\phi_0\rangle \neq 0,$$

often taken as definition in the literature.

4. THE COMPUTATION OF DOUBLE LIMIT POINTS

Like in the spherical shell problem (Section 2), we now consider a nonlinear equation depending on two real parameters. Keeping the same notation as in the preceding section, we write $G(u, \lambda, \mu) = 0$, where $G: \mathfrak{B} \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{B}$. For fixed $\mu = \bar{\mu}$, assume that G has a limit point with respect to λ, that is, $(u_0, \lambda_0, \bar{\mu})$ is a limit point in accordance with Definition 2. Consider the extended system

$$\left.\begin{aligned} G(u, \lambda, \mu) &= 0 \\ G_u(u, \lambda, \mu)h &= 0 \\ l(h) - 1 &= 0 \end{aligned}\right\} F(v, \mu) = 0, \quad v := (u, h, \lambda). \tag{6}$$

Suppose $(u_0, \lambda_0, \mu_{0'})$ is a non simple limit point of G with respect to λ, then Theorem B implies

(a′) $\operatorname{Ker} \hat{G}_v^0 = \operatorname{Ker} F_v^0 = \{\alpha\Phi_0 \mid \alpha \in \mathcal{R}\}$, $\Phi_0 \in Y = \mathcal{B} \times \mathcal{B} \times \mathcal{R}$

(b′) $\operatorname{Range} F_v^0 = \{y \in Y \mid \langle \Psi_0, y\rangle = 0\}$ where $\Phi_0 = (\phi_0, u_1, 0)$, $\Psi_0 \in Y'$.

Hence F_v^{-1} does not exist at $(v, \mu) = (v_0, \mu_0)$, with $v_0 = (u_0, \phi_0, \lambda_0)$, consequently (v_0, μ_0) must be a singular point of $F(v, \mu) = 0$. In order to guarantee that F has a limit point with respect to μ, we assume

$$(c')\ F_\mu^0 \notin \operatorname{Range} F_v^0, \qquad F_v^0 = F_v(v_0, \mu_0).$$

The following theorem [8] reduces the computation of a double limit point of G to that of a simple limit point of F.

THEOREM C. Assume that condition (c') holds. Then a double limit point (u_0, λ_0, μ_0) of G with respect to λ corresponds to a simple limit point (v_0, μ_0) of F with respect to μ.

Thus, in order to compute a double limit point of G, we have to solve the extended system of $F(v, \mu) = 0$, that is,

$$\left.\begin{aligned} F(v, \mu) &= 0 \\ F_v(v, \mu)k &= 0 \\ m(k) - 1 &= 0 \end{aligned}\right\} \hat{F}(v, k, \mu) = 0 \qquad v, k \in Y \tag{7}$$

where m is functional that ensures $k \neq 0$, for instance $\langle m, k\rangle$, $m \in Y'$. According to Theorems A and C, $\hat{F}_z^{-1}$ exists at $(v, \mu) = (v_0, \mu_0)$, where $z := (v, k, \mu)$, implying that the solutions of $\hat{F}(v, k, \mu) = 0$ are isolated, so that the Newton method is applicable.

It was shown in [12] that eqn (7) can be simplified by returning to the original notation. In fact, a short calculation reduces eqn (7) to

$$\begin{aligned} &G(u, \lambda, \mu) = 0, \qquad G_u(u, \lambda, \mu)h = 0, \\ &G_u(u, \lambda, \mu)w = -G_{uu}(u, \lambda, \mu)hh, \\ &l(h) = 1, \qquad l'(h)w = 0, \end{aligned} \tag{8}$$

which is a system of five equations in the five unknowns $u, h, w \in \mathcal{B}$ and $\lambda, \mu \in \mathcal{R}$. If $\langle l, h\rangle = 1$, one has $\langle l, w\rangle = 0$. In [8], the system (7) was solved by a method of false position. However, in view of Theorem C, we have found it convenient to solve eqn (8) by the Newton method as in the case of simple limit points. For this we need sufficiently close starting values for both h and w in (8). This can be done precisely as in the previous section. At a regular point $(\bar{u}, \bar{\lambda}, \bar{\mu})$, one first solves $G_u h = c^*$ as in eqn (4). The solution h^* is then scaled to satisfy $l(h) = 1$, which gives h_0 and $c = \alpha c^*$ according to eqn (5). With this, $w = w^*$ is computed from eqn (8) without the restriction $l'(h)w = 0$. If $(\bar{u}, \bar{\lambda}, \bar{\mu})$ is approaching a double limit point, $c \to 0$ and $l'(h)w^* \to 0$.

5. NUMERICAL IMPLEMENTATION: ODE

A computational procedure will now be described to solve the extended systems for the case that $G(u, \lambda) = 0$ stands for a system of ordinary differential equations (ODE) on a finite interval $I = [a, b]$, with two-point boundary conditions. An example is the axisymmetric deformation of shells of revolution [13, 14], where $u = [f(x), g(x)]$, f is a deflection, g a stress function, λ is a load parameter and μ is a shell curvature parameter.

Let the boundary value problem (BVP) be formally written as

$$Lu = N(x, u, \lambda, \mu), \quad x \in I, \quad Bu = 0, \quad x = a, b \tag{9}$$

where L, B are linear differential operators and N is in general nonlinear in u, λ and μ. The extended systems will be formulated in such a way that standard software for solving ODE-BVPs is directly applicable. A convenient package is COLSYS [15], which uses spline collocation in conjunction with the Newton method. It also allows for regular singular points at $x = a, b$ that occur, for instance, in the Reissner shell equations [13, 14].

Consider first simple limit points governed by eqn (2). Taking $l(h) = \| h \|_2$, we have from (2), for a fixed μ,

$$\begin{aligned} Lu &= N(x, u, \lambda, \mu) & Bu &= 0 \\ Lh &= N_u(x, u, \lambda, \mu)h & Bh &= 0 \\ y' &= h^T h = \sum_{i=1}^{k} h_i^2 & y(a) &= 0 \\ \lambda' &= 0 & y(b) &= 1 \end{aligned} \tag{10}$$

N_u is the matrix $(\partial N_i/\partial u_j)$, $h = [h_1(x), \ldots, h_k(x)]$. If m is the order of L ($m = 2$ for the axisymmetric shell problem), eqns (10) represent a BVP of order $2m + 2$ in the unknowns u, h, y and λ. Different choices for $l(h)$ are discussed in [11] and [12].

In order to obtain starting values for h, one simply replaces $\lambda' = 0$ in eqn (10) by $c' = 0$ and modifies the second equation of (10) to

$$Lh = N_u(x, u, \bar{\lambda}, \mu)h + ce \tag{11}$$

where e is an arbitrary constant unit vector. The three equations for $y(x)$ then imply that c satisfies (5).

Turning to the computation of double limit points, the first two equations of eqn (8) together with $l(h) = 1$ are identical with eqn (10), except that μ is an additional variable. The remaining equations of (8) are, for the present case,

$$\begin{aligned} &Lw - N_u(x, u, \lambda, \mu)w = -N_{uu}(x, u, \lambda, \mu)h \otimes h \\ &z' = h^T w = \sum_{i=1}^{k} h_i w_i, \qquad Bw = 0 \\ &\mu' = 0, \qquad z(a) = z(b) = 0. \end{aligned} \tag{12}$$

N_{uu} is a third order tensor, that is, the right-hand side of the first equation of (12) reads

$$(N_{uu} h \otimes h)_i = \sum_{j,r} \frac{\partial^2 N_i}{\partial u_j \partial u_r} h_j h_r, \qquad i = 1, \ldots, k.$$

The extended system (10), (12) for double limit points represents a BVP of order $3m + 4$ in the unknowns u, h, w, y, z, λ and μ. It is obvious how to modify the system to obtain the starting values for h and w.

Remark. It has been assumed here, for simplicity, that N is not a differential operator with respect to u. However, all of the above equations can easily be extended to this case by computing appropriate Fréchet derivatives; an example is given in Section 9.

6. NUMERICAL IMPLEMENTATION: PDE

Let $G(u, \lambda) = 0$ denote a system of nonlinear partial differential equations (PDE) on a finite domain $D \subseteq \Re^N$, with linear homogeneous conditions on the boundary ∂D

of D. Again, treating for simplicity a special case, let $u = u(x_1, \ldots, x_N)$ be a scalar function, and let the BVP be given by

$$Lu = \lambda f(u, \mu) \qquad x \in D, \qquad Bu = 0 \qquad x \in \partial D. \tag{13}$$

L is a linear elliptic operator, f a nonlinear function of u and μ, and B a boundary operator compatible with L. An example treated frequently in the literature is the thermal ignition problem

$$\Delta u + \lambda \exp[u/(1 + \mu u)] = 0 \qquad x \in D, \qquad u = 0 \qquad x \in \partial D.$$

Both simple and double limit points occur in this problem. The system of shallow shell PDEs represents another but more complex example of particular interest, for which we expect to report results in a sequel to this paper.

The extended system (2) is given by

$$\left.\begin{array}{ll} Lu = \lambda f(u, \mu) & \\ Lh = \lambda f_u(u, \mu)h & x \in D \\ l(h) = 1. & \end{array}\right\} \left.\begin{array}{l} Bu = 0 \\ Bh = 0 \end{array}\right\} x \in \partial D. \tag{14}$$

Two simple choices of l used in what follows are

$$l(h) = \int_D h \, \mathrm{d}x, \qquad l(h) = h(x_M) \qquad x_M \in D \tag{15}$$

where x_M is a point sufficiently distant from ∂D, for instance the midpoint when D is symmetric.

In order to solve eqn (14) by Newton's method, let us first determine starting values for h by solving the linear inhomogeneous BVP

$$Lh^* = \bar{\lambda} f_u(u, \bar{\mu})h^* \qquad x \in D, \qquad Bh^* = c^* \qquad x \in \partial D \tag{16}$$

with a constant $c^* \neq 0$. Here $(\bar{u}, \bar{\lambda})$ is a regular solution, $\bar{\mu}$ remains fixed along the solution branch in question. The desired starting value h is obtained by the scaling $l(\alpha h^*) = 1$ as in eqn (5). The Newton iteration for eqn (14) is, with $u_{n+1} = u_n + U_n$, $h_{n+1} = h_n + H_n$, $\lambda_{n+1} = \lambda_n + \Lambda_n$, $f' := f_u$ and suppressing the dependence of f on μ,

$$\begin{gathered} LU_n - \lambda_n f'(u_n)U_n - f(u_n)\Lambda_n = r_n \\ LH_n - \lambda_n f'(u_n)H_n - \lambda_n f''(u_n)h_n U_n - f'(u_n)h_n\Lambda_n = s_n \\ r_n := \lambda_n f(u_n) - Lu_n, \qquad s_n := \lambda_n f'(u_n)h_n - Lh_n \\ BU_n = BH_n = 0 \qquad \text{on} \qquad \partial D \\ l(H_n) = 0, \quad \text{provided} \quad l(h_0) = 1. \end{gathered} \tag{17}$$

In order to solve the linear system (16) and (17) numerically, D is either replaced by a grid F_N of N interior mesh points P_i, or D is subdivided into N finite elements. In the former case, all functions u_n, U_n etc. are restricted to F_N, and any standard finite-difference apporximation for L can be used for the terms LU_n, LH_n. Let $\mathbf{z}$ be the column vector of $h^*(P_i)$, $P_i \in F_N$, $i = 1, \ldots, N$, then the discretized eqn (16) can be written as a linear system $A\mathbf{z} = \mathbf{r}$, with a symmetric $N \times N$-matrix A (if L is the Laplace operator, A is block-tridiagonal). Standard methods and routines for the fast solution of such systems are available.

The simple structure of A is, of course, destroyed in the discretized eqn (17), when written as a linear algebraic system $C\mathbf{z} = \mathbf{s}$, with

$$\mathbf{z} := [U_n(P_1), H_n(P_1), U_n(P_2), H_n(P_2), \ldots, U_n(P_N), H_n(P_N), \Lambda_n]^T.$$

As before, $P_i \in F_N$, $i = 1, \ldots, N$, hence both $\mathbf{z}$ and $\mathbf{s}$ are $2N + 1$ vectors. The system $C\mathbf{z} = \mathbf{s}$ of eqn (17) then has the following structure [10]

$$\begin{pmatrix} A & \mathbf{a} \\ \mathbf{c}^T & \beta \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \xi \end{pmatrix} = \begin{pmatrix} \mathbf{r} \\ \rho \end{pmatrix} \tag{18}$$

where A is a $K \times K$ matrix, $K = 2N$, $\mathbf{a}$, $\mathbf{c}$, $\mathbf{r}$ and $\mathbf{x}$ are K vectors and β, ρ and ξ are scalars. The column $\mathbf{a}$ represents the coefficients of the unknown $\xi = \Lambda_n$, the row $\mathbf{c}^T$ is either the discretized integral eqn (15) or simply the condition $H_n(x_M) = 0$ implied by eqn (15). Matrices as in eqn (18) are known as bordered matrices, there is a well-known algorithm to solve eqn (18) by merely inverting A. However, A is singular at the limit point. Although Newton's method can still be applied, the convergence is no longer quadratic near the singularity, which has been observed in numerical calculations for the thermal ignition problem mentioned earlier.

In order to obtain an algorithm that converges quadratically, we first reorder the variables in the vector $\mathbf{z}$ and rewrite eqn (18) in the form

$$\begin{pmatrix} A' & O & \mathbf{a}_1 \\ D & A' & \mathbf{a}_2 \\ \mathrm{O} & \mathbf{c}_2^T & O \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \xi \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ O \end{pmatrix} \tag{19}$$

where $\mathbf{x}_1^T = (U_n(P_1), \ldots, U_n(P_N))$, $\mathbf{x}_2^T = (H_n(P_1), \ldots, H_n(P_N))$ and $\xi = \Lambda$. Here A' is a banded sparse $N \times N$ matrix obtained by discretizing the operator $L - \lambda_n f'(u_n)$. If $L = \Delta$, A' is a standard block-tridiagonal matrix. The term $-\lambda_n f''(u_n)h_n$ in eqn (17) transforms into the diagonal matrix D. Furthermore, we rearrange $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{r}$ in accordance with the splitting of $\mathbf{x}$ into $\mathbf{x}_1$ and $\mathbf{x}_2$. As A' is still singular at a limit point, the algorithm for bordered matrices cannot be applied to eqn (19). Now let A^* denote the $(N - 1) \times (N - 1)$ matrix obtained from A' by cancelling the last (or first) column $\mathbf{u}$ and row $\mathbf{v}^T$. Reducing D in the same way, denoting $(N - 1)$ vectors obtained from $\mathbf{a}_1$, etc. by cancelling the last (or first) component by $\mathbf{a}_1^*$, etc. and exchanging columns and rows, we can rewrite eqn (19) in the form

$$\begin{pmatrix} A^* & \mathrm{O} & \mathbf{u}^* & \mathrm{O} & \mathbf{a}_1^* \\ D^* & A^* & \mathbf{O} & \mathbf{u}^* & \mathbf{a}_2^* \\ \mathbf{v}^{*T} & \mathbf{O} & \alpha & \mathrm{O} & \alpha_1 \\ \mathbf{O} & \mathbf{v}^{*T} & \delta & \alpha & \alpha_2 \\ \mathrm{O} & \mathbf{c}_2^{*T} & \mathrm{O} & \gamma & \mathrm{O} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_1^* \\ \mathbf{x}_2^* \\ \xi_1 \\ \xi_2 \\ \xi \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1^* \\ \mathbf{r}_2^* \\ \rho_1 \\ \rho_2 \\ \mathrm{O} \end{pmatrix} \tag{20}$$

where

$$\mathbf{x}_i = \begin{pmatrix} \mathbf{x}_i^* \\ \xi_i \end{pmatrix}, \mathbf{a}_i = \begin{pmatrix} \mathbf{a}_i^* \\ \alpha_i \end{pmatrix}, \mathbf{r}_i = \begin{pmatrix} \mathbf{r}_i^* \\ \rho_i \end{pmatrix}, \mathbf{u} = \begin{pmatrix} \mathbf{u}^* \\ \alpha \end{pmatrix}, \mathbf{v} = \begin{pmatrix} \mathbf{v}^* \\ \alpha \end{pmatrix} \tag{21}$$

for $i = 1,2$ and $\mathbf{c}_2^T = (\mathbf{c}_2^{*T}, \gamma)$. α is the element A'_{NN} (or A'_{11}) of the matrix A', δ is the last (or first) element of D. Note that all starred vectors and $\mathbf{O}$ are $(N - 1)$ dimensional. At simple limit points, A^* can be assumed to be nonsingular. In fact, it can be proved for special second-order equations, but there is a heuristic argument as well as convincing numerical evidence in more general cases.

The matrix of the system (20) may be called a threefold bordered matrix. The algorithm for simply bordered matrices can be generalized to this system as follows. First solve the subsystems

$$\begin{pmatrix} A^* & \mathrm{O} \\ D^* & A^* \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} \quad \text{equivalent to} \quad \begin{matrix} A^*\mathbf{p} = \mathbf{e} \\ A^*\mathbf{q} = \mathbf{f} - D^*\mathbf{e}, \end{matrix} \tag{22}$$

setting the right-hand sides $\mathbf{e}$, $\mathbf{f}$ equal to

$$\begin{pmatrix}\mathbf{u}^* \\ \mathbf{O}\end{pmatrix}, \begin{pmatrix}\mathbf{O} \\ \mathbf{u}^*\end{pmatrix}, \begin{pmatrix}\mathbf{a}_1^* \\ \mathbf{a}_2^*\end{pmatrix}, \begin{pmatrix}\mathbf{r}_1^* \\ \mathbf{r}_2^*\end{pmatrix}. \tag{23}$$

Denote the solutions $\mathbf{p}$ and $\mathbf{q}$ of eqn (22) by $\mathbf{p}_i$, $\mathbf{q}_i$ and $i = 1, 2, 3, 4$ in the order of the sequence eqn (23), and form a 3×3 system from the lower right block of the matrix in eqn (20) and ρ_1, ρ_2, in order to find ξ_1, ξ_2 and $\xi = \xi_3$. This system can be reduced to

$$\begin{aligned}(\alpha - s_1)\,\xi_1 - \quad\quad s_2\xi_2 + (\alpha_1 - s_3)\,\xi_3 &= \rho_1 - s_4 \\ (\delta - t_1)\,\xi_1 + (\alpha - t_2)\,\xi_2 + (\alpha_2 - t_3)\,\xi_3 &= \rho_2 - t_4 \\ -\tau_1\xi_1 + (\gamma - \tau_2)\,\xi_2 \quad\quad - \tau_3\xi_3 &= -\tau_4\end{aligned} \tag{24}$$

where

$$s_i = \mathbf{v}^{*T}\mathbf{p}_i, \quad t_i = \mathbf{v}^{*T}\mathbf{q}_i, \quad \tau_i = \mathbf{c}_2^{*T}\mathbf{q}_i \qquad i = 1, \ldots, 4.$$

Finally, the solution $\mathbf{x}_1$ and $\mathbf{x}_2$ is obtained from

$$\mathbf{x}_1 = \mathbf{p}_4 - \sum_{i=1}^{3} \xi_i\mathbf{p}_i \qquad \mathbf{x}_2 = \mathbf{q}_4 - \sum_{i=1}^{3} \xi_i\mathbf{q}_i. \tag{25}$$

Note that, apart from the simple 3×3 system eqn (24), the only matrix to be inverted is A^*, which has the same structure as A', except that it is regular. Hence, a banded Gaussian elimination routine with partial pivoting can be used to solve the systems in step (22) and (23) of this algorithm. For large N, this step may also be carried out by an SOR-iteration or one of the more recent fast linear systems solvers (e.g. multigrid).

The extended system for double limit points can be discretized in much the same way. According to eqn (8) the only additional equations are $G_u(u, \lambda, \mu)w = -G_{uu}hh$ and $1'(h)w = 0$. Hence, in addition to $\mathbf{x}_1$, $\mathbf{x}_2$ and ξ, we have in the Newton linear system a vector $\mathbf{x}_3 = (W(P_1), \ldots, W(P)_N))$ and a scalar $\eta = M_u$, with $\mu_{n+1} = \mu_n + M_n$. The * operation then leads to a five-fold bordered matrix, where again A^* is the only large matrix to be inverted (for details see [12]).

7. ON THE COMPUTATION OF SIMPLE BIFURCATION POINTS

In this section we consider briefly a different type of singularity of an equation $G(u, \lambda) = 0$.

Definition 4. A singular point $(u_0, \lambda_0) \in S_G$ is called a simple bifurcation point if the conditions (a), (b) of Definition 2 are satisfied and if (c) is replaced by

$$\text{(d)}\quad G_\lambda(u_0, \lambda_0) \in \text{Range } G_u(u_0, \lambda_0),$$

which can be written equivalently as $\langle\psi_0, G_\lambda(u_0, \lambda_0)\rangle = 0$, with ψ_0 as in Definition 2. In the special case $G(0, \lambda) = 0$ for all $\lambda \in \mathfrak{R}$, a point $(0, \lambda_0)$ on the trivial solution branch $(0, \lambda)$ that satisfies (a), (b), and (d) is called a primary bifurcation point.

The numerical computation of bifurcation points is generally a much more difficult problem than the computation of limit points. An extended system similar to eqn (2) incorporating condition (d) is

$$\begin{aligned} G(u, \lambda) &= 0, \qquad G_u(u, \lambda)'\,\psi = 0 \\ \langle\psi, G_\lambda(u, \lambda)\rangle &= 0, \qquad \langle\psi, k\rangle - 1 = 0 \end{aligned} \tag{26}$$

where $k \in \mathfrak{B}$ is chosen to scale ψ. This system in the unknowns u, ψ and λ is over-determined. Special techniques for solving eqn (26) have been developed; some use the generalized inverse L^+ of a linear operator L, others use convex optimization.

In [11], the extended system (2) was used also for computing bifurcation points. In view of Theorem A, its solutions are not isolated in general. However, there are

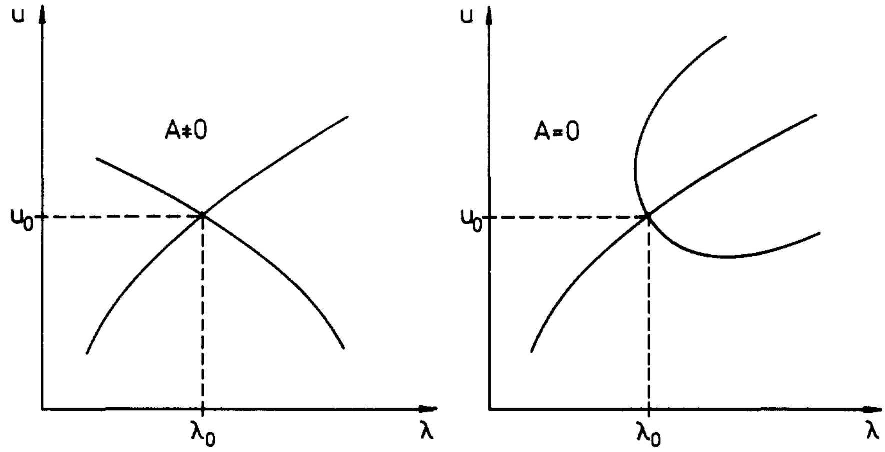

Fig. 3. Simple bifurcation points in the cases $A \neq 0$ and $A = 0$ (pitchfork bifurcation point).

some specific cases for which isolateness can be recovered. One case of particular interest in elastic stability is treated in Sections 8 and 9. It will be convenient to quote here a well-known result from bifurcation theory (see, for example [16]).

THEOREM D. Suppose (u_0, λ_0) is a simple bifurcation point of $G(u, \lambda) = 0$, and let the constants A, B, C be defined by

$$\begin{aligned} A &:= \langle \psi_0, G^0_{uu}\phi_0\phi_0 \rangle \qquad B := \langle \psi_0, G^0_{u\lambda}\phi_0 + G^0_{uu}\phi_0 w_0 \rangle \\ C &:= \langle \psi_0, G^0_{\lambda\lambda} + 2G^0_{u\lambda} w_0 + G^0_{uu} w_0 w_0 \rangle \\ w_0 &:= \text{solution of } G^0_u w_0 = -G^0_\lambda, \qquad \langle \psi_0, w_0 \rangle = 0, \end{aligned} \tag{27}$$

then the solutions of $G(u, \lambda) = 0$ in a neighborhood of (u_0, λ_0) can be written in the form

$$u = u_0 + \alpha\phi_0 + v(\alpha, \beta), \quad \lambda = \lambda_0 + \beta, \quad v(0, 0) = 0 \tag{28}$$

where v is uniquely determined for $|\alpha| \leq \alpha_0$, $|\beta| \leq \beta_0$, α_0 and β_0 positive, and where the "bifurcation equation"

$$g(\alpha, \beta) := \langle \psi_0, G(u_0 + \alpha\phi_0 + v(\alpha, \beta), \lambda_0 + \beta) \rangle = 0 \tag{29}$$

determines the relation between α and β. The function g has the properties

$$g(0, 0) = g_\alpha(0, 0) = g_\beta(0, 0) = 0$$
$$g_{\alpha\alpha}(0, 0) = A, \quad g_{\alpha\beta}(0, 0) = B \quad g_{\beta\beta}(0, 0) = C.$$

Moreover, the solutions near (u_0, λ_0) consist of two branches which intersect transversely at (u_0, λ_0) as indicated in Fig. 3. For $A \neq 0$ both branches, for $A = 0$ only one branch can be parameterized by λ. The second branch in the case $A = 0$ can be parameterized by $\pm(\lambda - \lambda_0)^{1/r}$ or $\pm(\lambda_0 - \lambda)^{1/r}$, where r is a positive integer $r > 1$ (e.g. $r = 2$ if $g_{\beta\beta\beta}(0, 0) \neq 0$). In terms of A, B and C, eqn (29) can be written as

$$A\alpha^2 + 2B\alpha\beta + C\beta^2 + g_1(\alpha, \beta) = 0, \quad g_1 = o[(|\alpha| + |\beta|)^2]. \tag{30}$$

Omitting the higher order terms represented by g_1, solutions of eqn (30) for sufficiently small α, β are in 1–1 correspondence to the solutions of $G(u, \lambda) = 0$ near (u_0, λ_0). We conclude this section with

Definition 5. A simple bifurcation point (u_0, λ_0) is called a pitchfork bifurcation point if $A = 0$, $B \neq 0$ (see Fig. 3).

8. THE COMPUTATION OF SYMMETRY-BREAKING BIFURCATION POINTS

Some of the numerical difficulties in the computation of bifurcation points disappear when bifurcating solutions do not inherit the symmetry of a primary solution branch. Some simple types of symmetry change have been discussed in [16] and [17]. It is assumed there that a symmetry $S \in L(\mathcal{B}, \mathcal{B})$ has the properties

$$S \neq I, \quad S^2 = I, \quad G(Su, \lambda) = SG(u, \lambda), \quad \mathcal{B} = B_s \oplus B_a \tag{31}$$

where $B_a = \{u \in \mathcal{B} \mid Su = -u\}$ and $B_s = \{u \in \mathcal{B} \mid Su = -u\}$. In this notation, symmetry breaking bifurcation is defined as follows.

Definition 6. A simple bifurcation point (u_0, λ_0) is called symmetry breaking if $u_0 \in B_s$ and $\phi_0 \in B_a$, where $\phi_0 \in \operatorname{Ker} G_u^0$, $\phi_0 \neq 0$.

For the computation of symmetry-breaking bifurcation points, we make use of the following basic theorem, proved in [17] for the above symmetry S.

THEOREM E. Let (u_0, λ_0) be a simple symmetry-breaking bifurcation point with $u_0 \in B_s$ and $\phi_0 \in B_a$. Then $A = 0$ is satisfied. Hence, if $B \neq 0$ also holds, (u_0, λ_0) is a pitchfork bifurcation point. Furthermore, if the extended system (2) is considered as a mapping $\hat{G}: Z \rightarrow Z$ with $Z := B_s \times B_a \times \mathfrak{R}$, then (u_0, h_0, λ_0) is an isolated solution of eqn (2) if and only if (u_0, λ_0) is a pitchfork bifurcation point.

An example of eqn (31) is $S = -I$, implying, $G(-u, \lambda) = -G(u, \lambda)$. Here $B_s = \{0\}$, $B_a = \mathcal{B}$, and bifurcation occurs from the trivial solution branch $(0, \lambda)$, a situation contained in the above theorem.

As a second example of symmetry breaking, consider the more concrete case where G stands for an elliptic BVP involving $u := u(r, \theta)$, with r, θ referring to polar coordinates. Suppose there exists a symmetric branch C_s of solutions (u, λ) where $u = u_0(r) \in B_s$. We then look for points (u_0, λ_0) on C_s where new branches C_a of asymmetric solutions $u = u(r, \theta) \in B_a$ intersect, for instance solutions of the type $u = U(r, \lambda) \cos n\theta + u_1$, with $u_1 \rightarrow u_0(r)$, $U \rightarrow 0$ as $\lambda \rightarrow \lambda_0$, n a positive integer. Although the symmetry is different here from that in eqn (31), some conclusions of Theorem E still hold. First, as symmetry breaking implies $A = 0$, it is sufficient to verify $B \neq 0$. Theorem E can then be applied, and the calculation of (u_0, λ_0) proceeds as in the case of simple limit points by solving the system

$$G(u, \lambda) = 0, \quad G_u(u, \lambda)h = 0, \quad l(h) = 1. \tag{32}$$

For pitchfork bifurcation points, the solution (u, h, λ) is isolated and vice versa. Note that in eqn (32) $u \in B_s$, $h \in B_a$, that is, $u = u(r)$, $h = h(r, \theta)$. The computational procedures for solving eqn (32) have been discussed in Sections 5 and 6. With appropriate changes due to $u \in B_s$ and $h \in B_a$, these procedures also apply to the computation of symmetry-breaking bifurcation points.

9. APPLICATION TO SPHERICAL SHELL PROBLEMS

The computation of symmetry-breaking bifurcation points will now be illustrated by the spherical shell buckling problem. Resuming the discussion of Section 2, consider a clamped shallow spherical shell under axisymmetric normal load $p(r)$. The basic equations can be written in the dimensionless form [2]

$$G(\mathbf{u}, \lambda, \mu) := \begin{cases} \Delta^2 f - \mu\Delta g - K[f, g] - \lambda p(r) = 0 \\ \Delta^2 g + \mu\Delta f + \frac{1}{2}K[f, f] = 0 \end{cases} \tag{33}$$

where

$$\Delta f = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta}, \quad K[f, g] := M[f, g] + N[f, g],$$

$$rM[f, g] := f_{rr}\left(g_r + \frac{1}{r}g_{\theta\theta}\right) + g_{rr}\left(f_r + \frac{1}{r}f_{\theta\theta}\right),$$

$$\frac{1}{2}r^2N[f; g] := \frac{1}{r}(f_{r\theta}g_\theta + f_\theta g_{r\theta}) - f_{r\theta}g_{r\theta} - \frac{1}{r^2}f_\theta g_\theta,$$

$$\mu = 2m^2\frac{H}{t}, \quad m^2 = [12(1 - \nu^2)]^{1/2},$$

and $\mathbf{u} = (f, g)^T$, with f and g denoting dimensionless normal displacement (w) and radial stress resultant (N_r), respectively. H is the shell rise, t the shell thickness, and λ is the dimensionless load intensity, with $|p(r)| \leq 1$. The boundary conditions are

$$\left.\begin{aligned} &f = f_r = 0, \quad g_{rr} - \nu(g_r + g_{\theta\theta}) = 0 \\ &g_{rrr} + g_{rr} - g_r + (\nu + 2)g_{r\theta\theta} - (\nu + 3)g_{\theta\theta} = 0. \end{aligned}\right\} \text{ at } r = 1 \tag{34}$$

The Fréchet derivative G_u, computed from eqn (33), is

$$G_u(\mathbf{u}, \lambda, \mu) = \begin{pmatrix} \Delta^2 - K[g,.], & -\mu\Delta - K[f,.] \\ \mu\Delta + K[f,.], & \Delta^2 \end{pmatrix} \tag{35}$$

In the following, we shall take G to include both eqns (33) and (34), but as the boundary conditions are linear, the part of G given by eqn (34) remains unchanged in the corresponding part of G_u. Obviously, we have from eqn (33)

$$G_\lambda = \begin{pmatrix} -p(r) \\ 0 \end{pmatrix} \quad \text{and} \quad G_{u\lambda} = 0. \tag{36}$$

Setting $\mathbf{h}_1 = (f_1, g_1)^T$, $\mathbf{h}_2 = (f_2, g_2)^T$, the second Fréchet derivative G_{uu} can be written as

$$G_{uu}(\mathbf{u}, \lambda, \mu)\mathbf{h}_1 \otimes \mathbf{h}_2 = \begin{pmatrix} -K[f_1, g_2] - K[f_2, g_1] \\ K[f_1, f_2] \end{pmatrix}. \tag{37}$$

The symmetry $K[f, g] = K[g, f]$ has been used in the above calculations. As the equations (33) are quadratic, we have $G_{uuu}(\mathbf{u}, \lambda, \mu) \equiv 0$.

Consider now a branch C_s of axisymmetric solutions $[f(r), g(r), \lambda]$ of $G(\mathbf{u}, \lambda, \mu) = 0$, for fixed μ. In this case, eqn (33) reduces to a pair of second order ODEs [2]. We wish to find symmetry-breaking bifurcation points on C_s. More precisely, we seek nontrivial solutions $\mathbf{h} = (\phi, \psi)^T$ satisfying $G_u(\mathbf{u}_0, \lambda_0, \mu)\mathbf{h} = \mathbf{0}$ for some $\lambda = \lambda_0$, where $\mathbf{u}_0 = \mathbf{u}_0(r) = (f, g)^T$ and $\mathbf{h} = \mathbf{h}_n(r) \cos n\theta$, n a positive integer. This can be written explicitly as

$$\begin{aligned} \Delta^2\phi - \mu\Delta\psi - K[g, \phi] - K[f, \psi] &= 0 \\ \Delta^2\psi + \mu\Delta\phi + K[f, \phi] &= 0 \end{aligned} \tag{38}$$

in accordance with eqn (35), together with the boundary conditions (34) applied to ϕ, ψ instead of f, g. The K-operators in eqn (38) reduce as follows

$$K[g, \phi] = M[g, \phi] = \frac{1}{r}\left[g''\left(\phi_r + \frac{1}{r}\phi_{\theta\theta}\right) + g'\phi_{rr}\right], \tag{39}$$

and similarly for $K[f, \psi]$ and $K[f, \phi]$. Writing $\phi = \phi_n(r)\cos n\theta$, $\psi = \psi_n(r)\cos n\theta$, eqns (38) reduce to a pair of fourth order ODEs for ϕ_n, ψ_n, which are given below. Suppose that eqn (38) and (36) (applied to ϕ and ψ) have a solution $\mathbf{h}^0 = (\phi^0, \psi^0)^T \neq \mathbf{0}$ at $(\mathbf{u}_0, \lambda_0)$, then condition (d) is satisfied. Indeed, setting $h^0 = [\phi_n^0(r), \psi_n^0(r)^T \cos n\theta]$, we find

$$\langle \mathbf{h}^0, G_\lambda^0 \rangle = \int_0^{2\pi} \int_0^1 [-p(r)\phi_n^0(r)\cos n\theta]\, r\, dr\, d\theta = 0,$$

using eqn (36), and hence $(\mathbf{u}_0, \lambda_0)$ is a bifurcation point according to Definition 4. Next we show that $A = 0$ of eqn (27) is satisfied. From eqn (37) we have

$$\begin{aligned}\langle \mathbf{h}^0, G_{uu}^0 \mathbf{h}^0 \otimes \mathbf{h}^0 \rangle &= \int_0^{2\pi} \int_0^1 (-2\phi^0 K[\phi^0, \psi^0] + \psi^0 K[\phi^0, \phi^0]) r\, dr\, d\theta \\ &= \int_0^{2\pi} \cos^3 n\theta\, d\theta \int_0^1 \Phi_1(r)\, dr + \int_0^{2\pi} \cos n\theta \sin^2 n\theta\, d\theta \int_0^1 \Phi_2(r)\, dr = 0 \end{aligned} \quad (40)$$

for some functions Φ_1, Φ_2 depending on $\phi_n^0(r)$, $\psi_n^0(r)$, but whose form is immaterial. Thus we have verified $A = 0$ for the type of symmetry breaking at hand. In Theorem E, this comes out as a consequence of symmetry breaking, provided the symmetry S is of the special form eqn (31) with $\mathcal{B} = B_s \oplus B_a$. In the present case we may define

$$B_s = B_0 := \{[f(r), g(r)]^T\}, \quad B_a = B_n := \{[f(r)\cos n\theta, g(r)\cos n\theta]^T\}.$$

Thus we have $G_u(\mathbf{u}_0, \lambda_0, \mu)\mathbf{h}^0 \in B_n$, but $G_{uu}(\mathbf{u}_0, \lambda_0, \mu)\mathbf{h}_0\mathbf{h}_0 \in B_0 \oplus B_{2n}$. In order to show that $(\mathbf{u}_0, \lambda_0)$ is a pitchfork bifurcation point, we need to verify $B \neq 0$ (Theorem E). In view of eqn (36), this condition reduces to

$$\langle \mathbf{h}^0, G_{uu}^0 \mathbf{h}^0 \otimes \mathbf{h}^1 \rangle \neq 0, \quad G_u^0 \mathbf{h}^1 = -G_\lambda^0, \quad \langle \mathbf{h}^0, \mathbf{h}^1 \rangle = 0. \quad (41)$$

The equations for $\mathbf{h}^1 =: (\phi^1, \psi^1)^T$ are,

$$\begin{aligned} &\Delta^2\phi^1 - K[g, \phi^1] - \mu\Delta\psi^1 - K[f, \psi^1] = p(r) \\ &\Delta^2\psi^1 + K[f, \phi^1] + \mu\Delta\phi^1 = 0 \end{aligned}$$

together with eqn (34), applied to ϕ^1, ψ^1. It follows that $\mathbf{h}^1 = \mathbf{h}_0^1(r) + \alpha\mathbf{h}^0 \in B_0 \oplus B_n$. The last equation of eqn (41) shows that $\alpha = 0$, since the integrand of $\langle \mathbf{h}_0^1(r), \mathbf{h}^0(r, \theta)\rangle$ is of the form $\psi(r)\cos n\theta$. The first equation of eqn (31) yields

$$B = \langle \mathbf{h}^0, G_{uu}^0 \mathbf{h}^0 \otimes \mathbf{h}_0^1 \rangle = \int_0^{2\pi} \cos^2 n\theta\, d\theta \int_0^1 \psi(r)\, dr \neq 0 \quad (42)$$

where $\psi(r)$ depends on ϕ_n^0, ψ_n^0 and $\mathbf{h}_0^1$, which are not known in explicit form. However, eqn (42) shows that $B \neq 0$ is possible and $(\mathbf{u}_0, \lambda_0)$ is then a pitchfork bifurcation point. In the computational scheme below, eqn (42) must be verified numerically. We note in passing that

$$C = \langle \mathbf{h}^0, G_{uu}^0 \mathbf{h}_0^1 \otimes \mathbf{h}_0^1 \rangle = 0$$

implies the saddle-point nature of the "bifurcation equation" observed in other elastic stability problems.

Returning to the computation of bifurcation points, we observe that in the extended system eqn (2) $\hat{G}(u, h, \lambda)$ can be considered as a mapping

$$\hat{G} : B_0 \times B_n \times \mathfrak{R} \rightarrow B_0 \times B_n \times \mathfrak{R} \quad (43)$$

where $G(\mathbf{u}, \lambda) = 0$ represents the axisymmetric BVP (33), (34) given explicitly by eqn (44) below, and $G_u(\mathbf{u}, \lambda)\mathbf{h} = 0$ is equivalent to eqn (38) and boundary conditions. It follows that $\mathbf{u} \in B_0$, $\mathbf{h} \in B_n$ implies $G_u(\mathbf{u}, \lambda)\mathbf{h} \in B_n$. By the same argument as in the proof of Theorem E it can be shown that a solution $(\mathbf{u}_0, \mathbf{h}_0, \lambda_0)$ of $\hat{G}(\mathbf{u}, \mathbf{h}, \lambda) = 0$ is isolated if and only if $(\mathbf{u}_0, \lambda_0)$ is a pitchfork bifurcation point. Hence the numerical verification of $B \neq 0$ is equivalent to quadratic convergence of the Newton method.

Following the numerical implementation given in Section 5, we write the extended system in such a form that standard software for ODEs is directly applicable. Equations (38) are reduced to a system of ODEs by separating the $\cos n\theta$ part from $\mathbf{h}$ as in [2], introducing dimensionless variables y, z, $\hat{y}$, $\hat{z}$. The result of these straightforward calculations is the following BVP for the computation of symmetry-breaking pitchfork bifurcation points for clamped spherical shells under arbitrary axisymmetric pressure:

$$\begin{gathered} Lf = -\mu g + \lambda Q(x) + fg, \quad Lg = \mu f - \tfrac{1}{2} f^2 \\ f'(0) = g'(0) = 0, \quad f(1) = g'(1) + (1 - \nu)g(1) = 0 \end{gathered} \tag{44}$$

$$\begin{aligned} L_n y &= \hat{y}, \quad L_n z = \hat{z}, \quad L_n := d^2/dx^2 + (2n + 1)/x(d/dx) \\ L_n \hat{y} &= -\mu\hat{z} + (K_n z)f' + f\hat{z}, \quad K_n := d/dx - n(n - 1)/x \\ L_n \hat{z} &= \mu\hat{y} + (K_n z)g' - (K_n y)f' + [g\hat{z} - f\hat{y}] \end{aligned} \tag{45}$$

$$\begin{gathered} y'(0) = z'(0) = \hat{y}'(0) = \hat{z}'(0) = 0, \quad z(1) = z'(1) = 0 \\ \hat{y}(1) - (1 + \nu)[y'(1) - n(n - 1)y(1)] = 0 \\ \hat{y}'(1) + n\hat{y}(1) - (1 + \nu)n^2[y'(1) + (n - 1)y(1)] = 0 \end{gathered}$$

$$\begin{gathered} \xi' = y^2 + z^2 + \hat{y}^2 + \hat{z}^2 \quad \xi(0) = 0, \quad \xi(1) = 1 \\ \lambda' = 0 \end{gathered} \tag{46}$$

where

$$L = d^2/dx^2 + (3/x)d/dx, \quad Q(x) = (4/x^2) \int_0^x p(s)s\, ds.$$

For computing starting values for y, z, $\hat{y}$, $\hat{z}$ at a regular point, λ is given, so $\lambda' = 0$ is dropped from eqn (47), the term in the brackets of eqn (45) is replaced by $[g\hat{z} - f\hat{y} + c]$ and $c' = 0$ is added to the above system. The BVP eqns (44)–(46) for the unknowns f, g, y, $\hat{y}$, z, $\hat{z}$, λ and ξ is of order 14.

Finally, we wish to obtain information on the stability of the asymmetric branch of solutions, which is related to the imperfection sensitivity of the structure [18]. In Fig. 1, the branches labeled D and E are stable ($n = 2$) and unstable ($n = 3$), respectively. According to the general theory [16], the locally nonsymmetric branch C_a near a pitchfork bifurcation point (u_0, λ_0) can be represented in the form

$$\begin{aligned} u &= u_0 + \alpha\phi_0 + v(\alpha), \quad \lambda = \lambda_0 + \xi(\alpha) \\ \xi(\alpha) &= -(D/6B)\alpha^2 + o(\alpha^2), \quad v(\alpha) = o(\alpha) \\ D &:= g_{\beta\beta\beta}(0, 0) = \langle \psi_0, G^0_{uuu}\phi_0\phi_0\phi_0 + 3G^0_{uu}\phi_0 w_1 \rangle \\ w_1 &:= \text{solution of } G^0_u w_1 = G^0_{uu}\phi_0\phi_0, \quad \langle \psi_0, w_1 \rangle = 0. \end{aligned} \tag{47}$$

Thus the sign of D/B decides on the stability of C_a. It appears then that imperfection sensitivity as introduced by Koiter [18] can be derived directly from the above representation.

For the spherical shell problem, $D/3$ reduces to $\langle\phi_0, G^0_{uu}\phi_0 w_1\rangle$, the equations for $\mathbf{w}_1 =: (V, W)^T$ are, in view of eqns (35), (37) and (38),

$$\begin{aligned} \Delta^2 V - \mu\Delta W - K[g, V] - K[f, W] &= -2K[\phi, \psi] \\ \Delta^2 W + \mu\Delta V + K[f, v] &= K[\phi, \phi] \end{aligned} \tag{48}$$

together with eqn (34) applied to V and W. Since $(\phi, \psi) = \mathbf{h} \in B_n$ and $\langle\phi_0, w_1\rangle = 0$, it follows that V and W have the form

$$A_1(r) + A_2(r)\cos 2n\theta, \quad \text{i.e.,} \quad \mathbf{w}_1 \in B_0 \oplus B_{2n}.$$

As in eqn (45), the BVP for V, W can be reduced to a system of ODEs by separation of variables. The resulting equations are equivalent to those derived in earlier work on imperfection sensitivity from Koiter's theory (see, for example [19]). We note that the solution of the linear system (48) and the calculation of D is straightforward; more details are found in [20].

10. NUMERICAL RESULTS

We present some examples from our calculations of simple and double axisymmetric limit points and of simple asymmetric (pitchfork) bifurcation points using the extended systems derived in the preceding sections. More extensive results for spherical caps will be found in [20] for a variety of different loads and boundary conditions, including some cases of nonuniform load, which have been discussed recently by Wan [21].

Table 1 shows examples of simple limit points for uniformly loaded shells, using the system (10) applied to Reissner's equations [13]. In contrast to the sketch of Fig. 2, a most noteworthy observation is that only very few regular solution points on the axisymmetric p–v-curve need to be computed in order to extract sufficiently close starting values for the extended system to converge to a limit point. Therefore, it becomes possible to calculate symmetrical buckling loads much more precisely than in [1]. In particular, only lower bounds for the buckling pressures $p_c = \lambda_0/\mu^2$ were given in [1] for $\mu \geq 100$, because a precise location of p_c presented considerable numerical difficulties due to the near-singularity of the system $G(u, \lambda) = 0$. With the present method, no convergence difficulties were encountered in computing accurate values of p_c. The number of regular solutions calculated before the singular solution was obtained is given in the tables. Clearly, more iterations in the solution of the extended system are needed if the program "jumps from a large distance" directly into the limit point. For a simply supported shell, $\mu = 100$, the lower bound given in [1] is 0.755, which is close to the value 0.75797 of Table 1. For a clamped shell, $\mu = 100$, the lower bound in [1] is 0.780, whereas the accurate value is 0.81425.

Table 1.

Clamped edge	Regular solutions λ	Simple limit points λ_0	$p_c = \lambda_0/\mu^2$	N
16	30, 60, 90, **120**	146.81	0.57349	4
49	500, 1000, **1500**	2554.3	1.06384	7
100	2000, 4000, 6000, **8000**	8142.5	0.81425	3
Simple support μ	λ	λ_0	$p_c = \lambda_0/\mu^2$	N
5	1, 2, **3**	10.956	0.43825	4
10	20, 30, **40**	48.996	0.48996	4
100	2000, 4000, **6000**	7579.7	0.75797	8

N = number of Newton iterations to get λ_0 from last λ (boldface).

Table 2.

	Regular solutions		Double limit points		
	μ	λ	λ_0	μ_0	N
Clamped	11	20, 40, **60**	76.929	10.891	5
Simple support	5	2.5, 5, **7.5**	9.626	4.544	6
Free edge	14	5, 10, **15**	24.969	12.807	5

N = number of Newton iterations to get λ_0 from last λ (boldface).

Table 3.

Clamped edge	Regular solutions			Nonsymmetric bifurcation points $p_n = \lambda_0/\mu^2$					
μ	λ			$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
36	200,	400,	**600**	0.77326	0.82750	0.93175			
49	400,	800,	**1200**	0.79326	0.75892	0.81139	0.90088		
81	1500,	2500,	**3500**		0.84515	0.77679	0.77656	0.81580	
100	2000,	4000,	**6000**			0.81263	0.77541	0.77935	0.81037

Table 2 shows examples of double limit points for shells under three different edge conditions using the system (12): a clamped edge, a simply supported and a free edge. For instance, starting with $\mu = 14$, free edge, and calculating only three regular points at $\lambda = 5$, 10 and 15, the solution of the extended system yields a double limit point at $\lambda_0 = 24.969$, $\mu_0 = 12.807$ after 5 iterations. Hence buckling of spherical caps with free edge disappears for $\mu \leq 12.807$. For these transition values μ_0 only very rough estimates were given in [1], extrapolated from several p–v-curves calculated for a discrete set of values of μ near μ_0. Here the saving of computing time when employing the present method is particularly striking.

Finally, we present some results for bifurcation points in Table 3. The observation is again that only few regular points on the axisymmetric solution branch were computed before switching to the solution of the extended system. In comparison to the computer work in [2], the saving is quite remarkable. After λ_0 is obtained for one value of n, the solution ($\mathbf{u}$, $\mathbf{h}$, λ_0) can be taken as starting value for computing λ_0 for $n + 1$ or $n - 1$. The numerical values of Table 3 for the nonsymmetric buckling pressures λ_0/μ^2 are generally in good agreement with those of [2], but they are more precise and much simpler to compute.

From the experience gained with the calculations discussed above (and in [20]), we may confidently conclude that stability problems of both shallow and nonshallow shells of general shape can be accurately analyzed by the methods presented here. In particular, the extension to shells of revolution presents no difficulties at all.

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